Introduction of Linear Programming Problem :-

Many business and economics situation are concerned with a problem of planning activity. In each case, there are limited resources at your disposal and your problem is to make such a use of these resources so as to yield the maximum production or minimize the cost of production or to give the maximum profit, etc. such problems are reffered to as the problems of constrained optimisation. Linear programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources. Programming is just another word for 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. The word linear stands for indicating that all relationships involved in a particular problem are linear.

In 1947, George Dentizig and his associates find out a technique for solving military planning problems while working in the U.S. department of Air Force. They observed that a large number of military programming and planning problems could be formulated as maximizing (minimizing) a linear forms of profit (cost) function whose variables are restricted to values satisfying a set of linear equations or inequalities.

Definition of Linear Programming Problem (LPP) :-

All organizations big or small, have at their disposal, men, machines, money and materials with limited supply. If the supply of these resources were unlimited there is a need for management tools like linear programming.

In the present, chapter some simple applications of linear programming problems and their mathematical formulations are discussed. The concepts are then extended to the general linear programming problem.

<u>Mathematical formulation of Linear programming problem</u>:- The term "linear" means that all the relations in the particular problem are linear & the term "programming" refers to the process determining a particular programme or plan of action.

A general linear programming problem includes a set of simultaneous linear equation which represent the conditions of the problem and a linear function which expresses the objective function of the problem.

The linear function which is to be optimized is called the objective function, and the conditions of the problem expressed as simultaneous linear equation are referred as constraints.

A general linear programming problem can be stated as <u>follows</u>:-

Find $x_1, x_2, \ldots, x_n$ which optimize the linear function $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ —— (1)

subject to the constraints

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n \ (\le, =, \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n \ (\le, =, \ge)\ b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n \ (\le, =, \ge)\ b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n \ (\le, =, \ge)\ b_m \end{array}\right] \quad (ii)$$

and non-negative restrictions

$$x_j \ge 0; \ j = 1, 2, \ldots, n \quad (iii)$$

Where all a_{ij}'s, b_i's and c_j's are constants & x_j's are variables.

In the condition given by (ii) there may be any of three signs $(\le, =, \ge)$.

The function z given by (i) is called the objective function and the conditions given by (ii) are termed as the constants of the linear programming problem.

We shall always assume that all $b_i \ge 0$. If any one is negative, we make it positive by multiplying both sides of the corresponding inequalities by -1. By this multiplication the inequality is also reversed.

The above linear programming problem may also stated in matrix form as:

Optimize $Z = C.x$

subject to $Ax \; (\leq, =, \geq) \; b$

and $x \geq 0$

Where $A = (a_{ij})$ is matrix of the coefficients of order $m \times n$.

$C = (c_1, c_2, \ldots, c_n)$ is a row vector known as price vector.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = [x_1, x_2, \ldots, x_n]^t$$ is column vector of variables.

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = (b_1, b_2, \ldots, b_m)^t$$

is column vector (is) called the requirement vector and 0 is n-dimensional null column vector.

The column vector formed by the coefficients of x_j in all the constraints is denoted by α_j.

i.e. $$\alpha_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = (a_{1j}, a_{2j}, \ldots, a_{mj})^t$$

then $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

<u>Note</u>:- In general linear programming problem it is assumed that the number of rows of coeff^t matrix A is less than its number of columns.

Assumption of Linear Programming Problem :-

There are one four basic assumptions which are necessary for linear programming problem :-

(I) Proportionality and Multiplicativity (linearity) :-

A primary requirement of linear programming is that the objective function and every constraint function must be linear. Also linearity implies the product of variables.

(II) Additivity :- Additivity means if a firm takes t_1 hours on machines G to make product A and t_2 hours to make product B, then the time on machine G devoted to produce A and B both is $t_1 + t_2$.

(III) Divisibility or continuity :- It means that the fractional levels of variables must be permissible besides integral values.

(IV) Deterministic or Certainty :- All the parameters in the linear programming model are assumed to be known exactly.

A practical problem which completely satisfied all the above assumption for L.P. is very rare indeed.

Example (1) :- A small manufacturer employs 5 skilled men and 10 remaining semi-skilled men and makes one article of two quantities a de luxe model and an ordinary model. The making of a defuse model requires 2 hours work by a skilled man 2 hours work by a semi-skilled man. The ordinary model requires 1 hours work. a skilled man and 3 hours work by a semi-skilled man. By union rules no man can work more than 8 hours per day. The manufacturers clear profit on the deluex model is Rs. 10 and on an ordinary model is Rs. 8 formulate the model of the problem. Do not solve it.

Solution :- Let the decision variables are x_1, x_2.

i.e. the number of deluxe models are manufactured $= x_1$

& the number of ordinary models are manufactured $= x_2$

Since profit on unit models of each of the above are Rs. 10 per deluxe model and Rs. 8 per ordinary model. therefore

Total profit $= 10x_1 + 8x_2$

Let z = Total profit $= 10x_1 + 8x_2$

Since one skilled labour works 8 hours per day

$\therefore$ He will work 5×8 hours in five days.

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Similarly; semi skilled labour will work 10x8 hours in five days.

∵ Unit deluxe manufacturing requires two hours by skill labour

∴ He will requires $2x_1$ hours by deluxe model and therefore,
Similarly, He will give $1x_2$ hours for the ordinary model.

the total time given by skill labour $= 2x_1 + x_2$

Since he has got 40 hours total time. therefore

$$2x_1 + x_2 \leq 40$$

Similarly, for the semi-skilled labour

$$2x_1 + 3x_2 \leq 80$$

Hence the mathematical modeling is

$$\text{Max } Z = 10x_1 + 8x_2$$

subject to constraints

$$2x_1 + x_2 \leq 40$$
$$2x_1 + 3x_2 \leq 80$$
$$x_1, x_2 \geq 0.$$

Example ②:— A lamp manufacturer produce two types of lamps A and B. Lamp A requires 2 hours of cutters times and 1 hours for finishers' time. and lamp B requires 1 hours of cutter time and 2 hours of finisher's time. Cutter has 104 hours and finisher 76 hours. Profit on one lamp of A is Rs 6.00 and on one B is Rs. 11.00 determine manufacturing policy of maximum returns.

Solution:— Let x_1 be no. of lamps of type A and x_2 be no. of lamps of type B. Since time requires of cutter's time: for type A per lamps is 2 hrs. Therefore $2x_1$ will be the time cutter's time for lamp A. Similarly $1 \cdot x_2$ will be the time of cutter's time for type of lamp B. Hence for total of cutter's time

$$= 2x_1 + x_2$$

Similarly, total time of finisher's time is 76 hours; for type A and B is given by

$$x_1 + 2x_2$$

So constraints are

$$2x_1 + x_2 \leq 104; \quad x_1 + 2x_2 \leq 76$$

Since profit of type A is Rs. 6 per lamp

total profit of type A $= 6x_1$

Similarly, for type B, profit $= 11x_2$

total profit on type A & B $= 6x_1 + 11x_2$

So problem is

$$\max.\ Z = 6x_1 + 11x_2$$

subject to the constraints

$$2x_1 + x_2 \leq 104$$
$$x_1 + 2x_2 \leq 76$$
$$\& \ x_1 \geq 0;\ x_2 \geq 0.$$

Solution of Linear Programming Problem :-

When a linear programming problem is formulated as a mathematical model, we will be wished to get the optimal solution of the problem. There are several methods to solve to different type of linear programming problems. Here we discuss some methods to solve the linear programming problem:

Geometrical or Graphical Method :-

A linear programming problem with two variables can be easily solved by using graphical or geometrical method. Graphical method provides a pictorial representation of the solution process and a great deal of insight into a basic concept used in solving large linear programming problem.

In this method we proceed as follows:

Step I :- Identify the problem, the decision variables the objective function, restrictions.

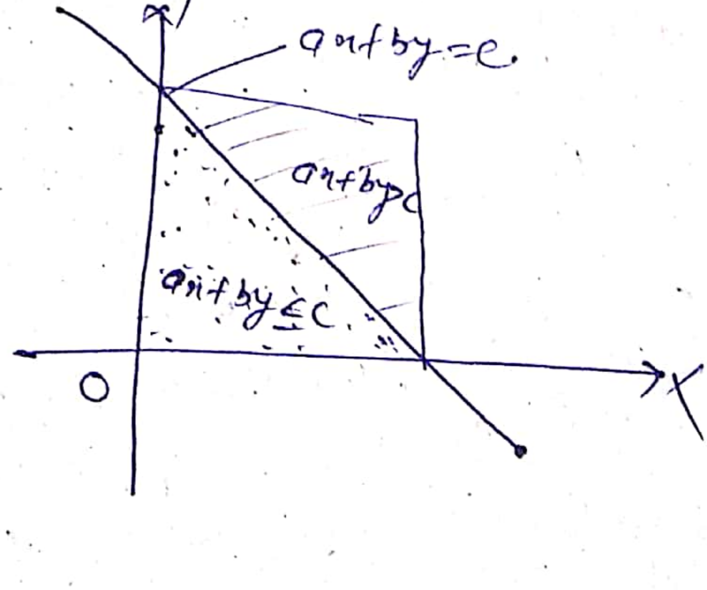
Step II :- Then we consider all the constraints as equalities.

Step III :- We draw the lines in the plane corresponding to each equation and non-negative restrictions.

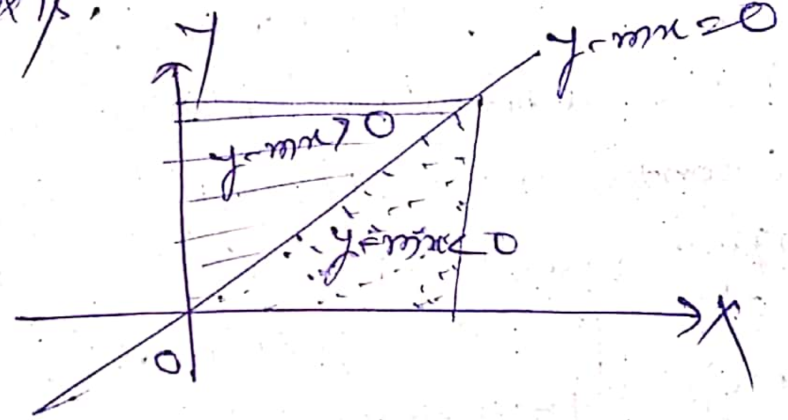
Method of draw lines :- Putting $x_1 = 0$ in the eqn of a line find x_2 and then putting $x_2 = 0$ find x_1. Thus, we get the point of inter-section of the line with the axes. The line is drawn by joining these points on the axis.

<u>Step IV</u> :— Then we find the permissible region (feasible region) for the value of the variables which is the region bounded by the lines drawn in step III. For which we proceed as follows:

Consider the constraint $ax+by \leq$ or $\geq c$, where $c>0$. The line $ax+by=c$ (drawn step III) divide in the xy-plane in two regions, one containing and the other not containing the origin. Since $(0,0)$ satisfy the inequality $ax+by \leq c$ (or $\geq$), so for the inequality $ax+by \leq c$, the feasible region is the region which contains the origin also, does not satisfy the inequality $ax+by \geq c$ so for the inequality $ax+by \geq c$, the feasible region is region which does not contain the origin.



Again consider the component $y - mx \leq$ or ≥ 0, where $m > 0$.
The line $y - mx = 0$ (drawn in step II) divide the xy-plane in two regions, one containing the +ve x-axis and other containing the +ve y-axis. For the inequality $y - mx < 0$, the feasible region is the region which contains the +ve x-axis and for the region $y - mx > 0$, the feasible region is the region which contains the +ve y-axis.



Thus, we find feasible regions corresponding to all inequalities. Then the region which is common to all these regions is the permissible regions (i.e. feasible region) for the values of variables. This permissible region is shaded.

<u>Step V</u> :- Here we find the point in the permissible region (obtained in step IV) which gives the optimum value of the objective function Z. The point will be one of the extreme point. (vertices) of the convex polygon enclosing the permissible region. The point, can be attained by the given further method:

Corner Point Method :- Determine the vertices of the convex polygon, which are the points of intersection of the straight lines passing through them. These vertices are called the extreme point of the set of all feasible solutions of the L.P.P. Then find the values of the objective function Z at all these points. The point where the objective function Z attains its optimum (maximum or minimum as the case may be) value gives the optimum or optimal value of the L.P.P.

If the two vertices of convex polygon given the same optimum value of the objective fun Z, then all points on the line segment joining these two vertices, will give the optimum value of the objective function Z. In these case the L.P.P. is said to be to have infinite number of optimum solution.

Iso-profit or Iso-cost method :- To find the vertex of the convex polygon, which gives the optimum value of the objective function Z, draw a straight line in the feasible region corresponding to the equation obtained by giving some convenient value to the objective function.

To obtain the extreme point of the permissible regions, giving the optimum value of the objective function Z, we go on drawing lines parallel to to the line Z = 0. The farthest extreme point is the vertex of the permissible region through which one of the parallel lines passes and after which it leaves the region and nearest extreme point is the vertex of the permissible region through which the parallel line enters the region.

Example ①: Solve the linear programming problem

$$\max. Z = 6x_1 + 11x_2$$

s.t. to

$$2x_1 + x_2 \le 104$$
$$x_1 + 2x_2 \le 76$$
$$x_1 \ge 0;\ x_2 \ge 0.$$

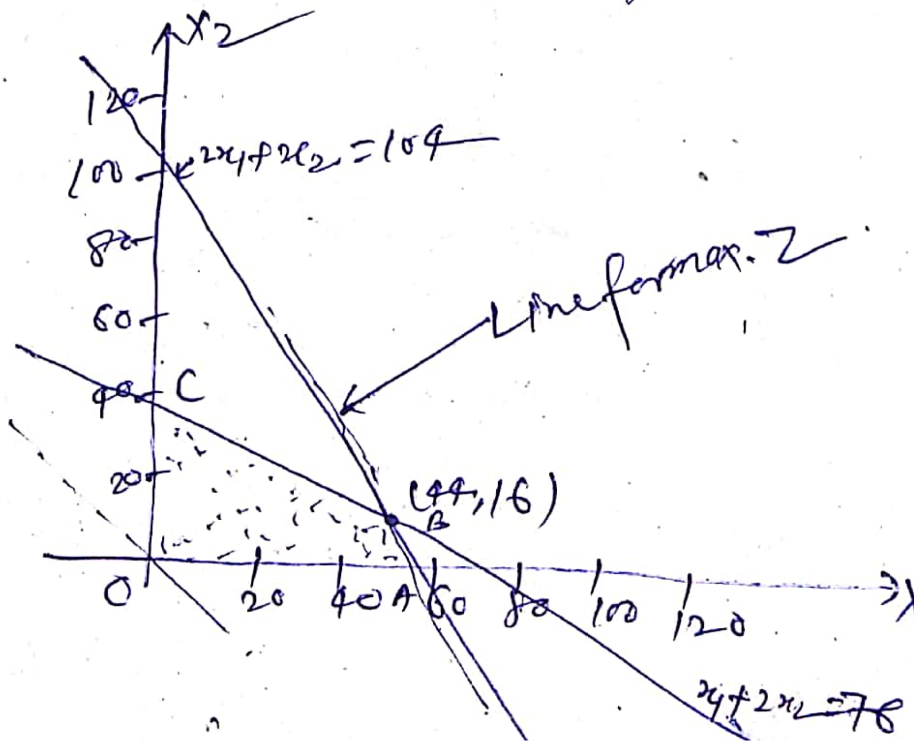
Solution:– We solve the given problem through following steps:

Step I: Here the given problem is of maximization & also inequalities and nonnegative restrictions hold.

Step II:– First of all convert the inequalities into equality; we get

$$2x_1 + x_2 = 104$$
$$x_1 + 2x_2 = 76$$

Step III:– Draw the line in the plane corresponding to each equation and non-negative restriction.



Step IV :- In this step we find the feasible region corresponding to all inequalities i.e the region which is common to all the regions. This feasible region OABC shaded.

Step V :- Now we find the point in feasible region obtained in step IV which gives the optimum value of the objective function Z.

To find such a point draw an iso profit or iso cost (dotted) line by giving some convenient value K to the objective function.

Then for the maximization problem the extreme point of the permissible region which is farthest away from this line gives the optimum value of Z.

Now, we move the iso profit line parallel to itself farther from origin. We observe that the iso profit line touches only one point B is termed as highest possible iso profit line. & point B gives the extreme point of feasible region.

Hence the optimum feasible solution obtained at B(44, 16) is

$$x_1 = 44, \quad x_2 = 16$$

with max. $Z = 6 \times 44 + 11 \times 16$

$= 264 + 176$

$=$ Rs. 440.

Example (2) :— Solve the programming problem:

$$\text{Max. } Z = 8x_1 + 7x_2$$

s. to
$$3x_1 + x_2 \le 66000$$
$$x_1 + x_2 \le 45000$$
$$x_1 \le 20000$$
$$x_2 \le 40000$$

and $x_1 \ge 0$, $x_2 \ge 0$.

Solution :— Step I :— The given problem is a maximization problem, inequalities and non-negative restrictions also satisfied.

Step II :— First we consider the constraints as equations, we get
$$3x_1 + x_2 = 66000$$
$$x_1 + x_2 = 45000$$
$$x_1 = 20000$$
$$x_2 = 40000$$

Step III :— Now we draw the lines to each of the above equations in two dimensional plane, as shown in figure on next page.

Step IV :— This step we find the region which is common to all the regions i.e. feasible region. Corresponding to all inequalities. The shaded region OABCDE is feasible region.

Step V :— Now we have to find the extreme point which gives the optimum value of objective function Z.

For this draw a dotted line by giving some convenient value K to the objective function. This dotted line is called Isoprofit line.

Now we move the isoprofit line parallel to itself farther from origin. Then we observe that the isoprofit line touches one and only one point C before leaving the feasible region. This point C is extreme point of the feasible region.

Hence, optimum feasible solution obtained at C(10500, 34500) is $x_1 = 10,500$, $x_2 = 34,500$

and Max. Z $= 8 \times 10500 + 7 \times 34500 =$ Rs. 325500.